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On the Existence of Invariant Measure for Lagrangian Velocity in Compressible Environments

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We study transport of a passive tracer particle in a time dependent turbulent flow in the medium with positive molecular diffusivity. We show that there exists then a probability measure equivalent to the underlying physical probability, corresponding to the Eulerian velocity field, under which the particle Lagrangian velocity observations are stationary. As an application we derive the existence of the Stokes drift and the effective diffusivity—the characteristics of the long time behavior of the particle motion.

KEY WORDS: Passive tracer; turbulent transport; Lagrangian dynamics; stationarity.

1. INTRODUCTION

The simplest model of the passive tracer motion in a turbulent flow is given by the Itô stochastic equation

$$\begin{cases} d\mathbf{x}(t) = \mathbf{u}(t, \mathbf{x}(t))\, dt + \sqrt{2\kappa}\, d\mathbf{w}(t), \\ \mathbf{x}(0) = \mathbf{0}, \end{cases} \tag{1.1}$$

where $\mathbf{u}: \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, the so-called *Eulerian velocity*, is a stationary, time-space strongly mixing, d-dimensional random field given over a certain probability triple $\mathscr{T}_0 := (\Omega, \mathscr{V}, \mathbb{P})$ and $\mathbf{w}(t)$, $t \geqslant 0$ is an independent of it standard d-dimensional Brownian motion, given over $\mathscr{T}_1 := (\Sigma, \mathscr{A}, W)$. The parameter $\kappa > 0$ characterizes the strength of the intrinsic molecular diffussivity of the medium.

Long time, large scale behavior of the tracer particle can be studied by considering the scaled trajectories

$$\varepsilon \mathbf{x}(t/\varepsilon^q) \text{ for a certain } q > 0 \text{ and a parameter } \varepsilon \ll 1. \tag{1.2}$$

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This problem displays a very rich phenomena of possible trajectory behaviors such as: Newtonian motions, diffusions, fractional diffusions and possibly Levy flights, see refs. 1–6. Most of the available rigorous results, e.g., those obtained by methods of homogenization theory, see ref. 7, are derived under incompressibility assumption on the drift, i.e., $\nabla_{\mathbf{x}} \cdot \mathbf{u}(t, \mathbf{x}) \equiv 0$. The fundamental property, which renders the incompressible case tractable, is the fact that the Lagrangian velocity process $\eta_t := \mathbf{u}(t, \mathbf{x}(t))$, $t \geqslant 0$, is stationary, as it is well known since the work of Lumley,[8] and also ref. 9. Stationarity of the Lagrangian process permits the extensive use of the methods of the ergodic theory of Markov processes, see e.g., refs. 7, 10, and 11.

In the compressible case this approach works when the invariant measure is known to exists, i.e., one can find a probability measure $\mathbb{P}_*$ on $(\Omega, \mathscr{V})$ such that

$$\mathbb{P}_* \otimes W[\eta_{t_1+h} \in A_1, \ldots, \eta_{t_N+h} \in A_N] = \mathbb{P}_* \otimes W[\eta_{t_1} \in A_1, \ldots, \eta_{t_N} \in A_N]$$

for any $0 \leqslant h$, $t_1 < \cdots < t_N$ and Borel measurable $A_1, \ldots, A_N \in \mathscr{B}(\mathbb{R}^d)$. $\mathbb{P}_*$ is called *regular* if, in addition, it is equivalent to $\mathbb{P}$, i.e., $\mathbb{P}[A] = 0$ iff $\mathbb{P}_*[A] = 0$ and ergodic. The latter means that

(E) for any Borel subset A of $C([0, +\infty); \mathbb{R}^d)$ satisfying

$$\mathbb{P}_* \otimes W[[\theta_h(\eta_\bullet) \in A] \triangle [\eta_\bullet \in A]] = 0$$

for any $h \geqslant 0$ we have $\mathbb{P}_* \otimes W[\eta_{\bullet+h} \in A] = 0$ or 1. Here $\triangle$ denotes the symmetric difference of events and θ_h is the canonical shift on $C([0, +\infty); \mathbb{R}^d)$.

When such a measure exists a simple application of the ergodic theorem yields the law of large numbers for the particle motion, i.e.,

$$\mathbf{v}_* := \lim_{t \uparrow +\infty} \frac{\mathbf{x}(t)}{t}$$

is a deterministic vector, also called the *Stokes dirft*. If one can show a sufficient decay rate of the correlations of the Lagrangian velocity then

$$\mathbf{D}_* := \lim_{t \uparrow +\infty} \frac{\mathbb{E}\mathbf{M}[(\mathbf{x}(t) - \mathbf{v}_* t) \otimes (\mathbf{x}(t) - \mathbf{v}_* t)]}{t}$$

also exists and is called an *effective diffusivity tensor* of the flow.[12] We say then that the tracer behavior is *asymptotically diffusive*. Here $\mathbb{E}$, $\mathbf{M}$ denote the means corresponding to both sources of randomness in (1.1).

Unfortunately, due to the infinite dimensional character of the problem, the existence of regular invariant measures is, in general, hard to prove except for some special cases, e.g. when the Eulerian velocity is a gradient of a steady (time independent) scalar potential field, i.e., $\mathbf{u}(\mathbf{x}) = \nabla_{\mathbf{x}}\phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$ and $\phi(\cdot)$ is stationary, with $Z = \int \exp\{-\phi(\mathbf{0})\}\, d\mathbb{P} < +\infty$. Then, the "Gibbs like" measure $\mathbb{P}_*(d\omega) := \exp\{-\phi(\mathbf{0}, \omega)\}\, d\mathbb{P}/Z$ is invariant, see e.g., ref. 7.

Most of the available results concerning motions in general compressible Eulerian velocity fields are limited to the case of *slowly varying fields*, i.e., $\mathbf{u}(t, \mathbf{x}) = \tilde{\mathbf{u}}(t, \varepsilon^r \mathbf{x})$, where $r > 0$ is coupled with q of (1.2), $\varepsilon \ll 1$, see e.g., refs. 13 and 14. The limit discounts then the spatial variation of the velocity, thus the analysis of the asymptotic behavior of trajectories can be then done by the use of perturbative techniques, avoiding in this way the problem of finding the invariant measure. In the case when the spatial and temporal variables are fully coupled, i.e., $r = 0$, the perturbation argument fails. However, it is generally believed that the particle should still display a diffusive behavior, provided that the Eulerian velocity field is sufficiently strongly decorrelating in time, see ref. 12.

In the present article we set out to prove the existence of regular invariant measures for Eulerian fields with fast temporal decorrelation properties, see Theorem 2.1 below. We apply further this result to prove the existence of the Stokes drift, Corollary 2.2. Finally, we establish, see Theorem 2.3, the diffusive character of the fluctuations of the trajectory around the mean motion. Namely, we show that the ratio of the mean square displacement of the particle, after discounting the mean motion, against time tends to a constant tensor (the so called *effective diffusivity of the medium*).

The organization of the paper is as follows. In Section 2 we introduce the notation and give precise formulation of the main results.

In Section 3 we give a construction of a regular invariant measure for the Lagrangian environment process. Speaking in loose terms it is the process that describes the environment viewed from the moving particle. The existence of such a measure implies immediately the existence of the Stokes drift, via the individual ergodic theorem.

The principal tool used in the course of the proof is an abstract operator acting on the space of probability densities with respect to measure $\mathbb{P}$—the physical probability describing the statistics of the environment. This operator, called here *the transport operator* is in some sense dual to a temporal shift over an interval greater than or equal to the correlation time of the Eulerian process, see (3.1). Yet, as it should be noted, the two functionals of the environment that appear in the duality relation must depend only on the future and the past of the Eulerian process correspondingly, cf. Proposition 3.1. This type of operator has appeared in the work of Komorowski and Papanicolaou[15] concerning the functional central limit theorem for motions in incompressible Eulerian fields.

The crucial observation is that the transport operator possesses a unique invariant density, see Lemma 3.2, which we use in the construction of the regular invariant measure for the Lagrangian process, see (3.5).

In the final Section 4 we present the proof of Theorem 2.3. The proof relies on the strong decorrelation properties of the transport operator, see Lemma 4.1.

2. NOTATION AND FORMULATION OF THE MAIN RESULT

Let (Ω, d) be a Polish space with a Borel probability measure $\mathbb{P}$. We denote by $\mathscr{B}(\Omega)$ the σ-algebra of Borel sets on Ω and by $\mathbb{E}[\,\cdot\,]$ the corresponding mathematical expectation. Let $\mathscr{N}$ be the σ-ring of $\mathbb{P}$-null sets in $\overline{\mathscr{B}(\Omega)}$, the completion of $\mathscr{B}(\Omega)$. Unless otherwise stated, we will assume, that any sub-σ-algebra of $\overline{\mathscr{B}(\Omega)}$ contains $\mathscr{N}$. For abbreviation sake we write $L^p := L^p(\mathscr{T}_0)$, where $\mathscr{T}_0 := (\Omega, \overline{\mathscr{B}(\Omega)}, \mathbb{P})$.

Let $\tau_{t,\mathbf{x}}: \Omega \to \Omega$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ be a group of measure preserving transformations, i.e., for any $t, s \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $A \in \overline{\mathscr{B}(\Omega)}$, $\tau_{t,\mathbf{x}} \circ \tau_{s,\mathbf{y}} = \tau_{t+s,\mathbf{x}+\mathbf{y}}$, $\tau_{t,\mathbf{x}}(A) \in \overline{\mathscr{B}(\Omega)}$ and $\mathbb{P} \circ \tau_{t,\mathbf{x}} = \mathbb{P}$. We suppose that $\mathbf{u}: \Omega \to \mathbb{R}^d$ is a random vector over $\mathscr{T}_0$ satisfying

$$\mathbb{E}\mathbf{u} = \mathbf{0}. \tag{2.1}$$

The Eulerian velocity field is defined as $\mathbf{u}(t, \mathbf{x}; \omega) := \mathbf{u}(\tau_{t,\mathbf{x}}(\omega))$. The field is time stationary, space homogeneous and the assumption (2.1) guarantees that it is of zero mean. We denote by $\mathscr{U}_a^b$, $-\infty \leqslant a \leqslant b \leqslant +\infty$ the σ-algebras generated by $\mathbf{u}(t, \mathbf{x})$, $a \leqslant t \leqslant b$, $\mathbf{x} \in \mathbb{R}^d$. We assume that

(FDT) *(finite decorrelation time)* there exists $T > 0$ such that for any $t \in \mathbb{R}$ the σ-algebras $\mathscr{U}_{-\infty}^t$ and $\mathscr{U}_{t+T}^{+\infty}$ are independent.

Finally, we suppose that the field possesses certain regularity both in the topological and measure theoretic sense. Namely we assume that

(R) the field $\mathbf{u}(\cdot,\cdot)$ is jointly Hölder continuous, of C^1 class in the spatial variable and there exists a deterministic constant $C > 0$ such that

$$\sup_{(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d} [|\mathbf{u}(t, \mathbf{x})| + |\nabla_{\mathbf{x}} \mathbf{u}(t, \mathbf{x})|] \leqslant C.$$

In addition, we suppose that all distributions of vectors $(\mathbf{u}(t_1, \mathbf{x}_1),\ldots, \mathbf{u}(t_N, \mathbf{x}_N))$, where $N \geqslant 1$, $(t_i, \mathbf{x}_i) \neq (t_j, \mathbf{x}_j)$, $i \neq j \in \{1,\ldots, N\}$, are absolutely continuous with respect to the Nd-dimensional Lebesgue measure.

Theorem 2.1. Suppose that $\mathbf{u}(\cdot,\cdot)$ is a velocity field satisfying (2.1), FDT, R and the trajectory $\mathbf{x}(t)$, $t \geqslant 0$ is given by (1.1) with $\kappa > 0$. Then, there exists a regular invariant probability measure $\mathbb{P}_*$ for the Lagrangian velocity process $\eta_t := \mathbf{u}(t, \mathbf{x}(t))$, $t \geqslant 0$.

As an immediate corollary from this theorem we obtain the following.

Corollary 2.2 (The Existence of the Stokes Drift). Under the assumptions of Theorem 2.1 the limit

$$\mathbf{v}_* = \lim_{t \uparrow +\infty} \frac{\mathbf{x}(t)}{t}$$

exists $\mathbb{P} \otimes W$ a.s. and is a deterministic vector equal to $\int \mathbf{u}(\omega)\, \mathbb{P}_*(d\omega)$.

In addition, we have also the following result.

Theorem 2.3 (The Existence of the Effective Diffusivity). Under the assumptions of Theorem 2.1 the limit

$$\mathbf{D}_* = \lim_{t \uparrow +\infty} \frac{1}{t}\, \mathbb{E} \otimes \mathbf{M}\{(\mathbf{x}(t) - \mathbf{v}_* t) \otimes (\mathbf{x}(t) - \mathbf{v}_* t)\}$$

exists. This matrix is called *the effective diffusivity of the medium.*

Remark. One can also show that the laws of the scaled trajectories

$$\tilde{\mathbf{x}}_\varepsilon(t) := \varepsilon \left[\mathbf{x}\left(\frac{t}{\varepsilon^2}\right) - \mathbf{v}_* \frac{t}{\varepsilon^2} \right], \qquad t \geqslant 0$$

over $C([0, +\infty); \mathbb{R}^d)$ converge weakly, as $\varepsilon \downarrow 0$, to a Wiener measure with the co-variance matrix $\mathbf{D}_*$. The proof of this fact can be routinely concluded from the estimates similar to those obtained in Section 4 but we shall not pursue this task in the present article. ∎

3. THE PROOF OF THEOREM 2.1

3.1. Transport Operator

According to Lemma A.1 of Appendix A below the conditions FDR and R together guarantee that $(\mathscr{U}^t_{-\infty})$ admits a factorization in the following sense. There exists a filtration of σ-algebras $\mathscr{R}^t$, $t \geqslant 0$ such that for any $t \geqslant 0$ we have $\mathscr{U}^t_{-\infty} = \mathscr{U}^0_{-\infty} \otimes \mathscr{R}^t$. The latter means that $\mathscr{R}^t$ is independent of

$\mathscr{U}^0_{-\infty}$ and they both generate $\mathscr{U}^t_{-\infty}$. In addition, there exists $T > 0$, for which $\mathscr{U}_T^{+\infty} \subset \mathscr{R}$, where $\mathscr{R} := \bigvee_{t=0}^{\infty} \mathscr{R}^t$. Let $\mathscr{T}_2 := (\Omega, \mathscr{U}^0_{-\infty}, \mathbb{P})$ and $\mathscr{T}_3 := (\Omega, \mathscr{R}, \mathbb{P})$. The aforementioned factorization implies the existence of a linear operator $Z: L^1(\mathscr{T}_0) \to L^1(\mathscr{T}_2 \otimes \mathscr{T}_3)$ such that

(Z1) $\int_\Omega \int_\Omega ZF(\omega, \omega')\, \mathbb{P}(d\omega)\, \mathbb{P}(d\omega') = \int_\Omega F(\omega)\, \mathbb{P}(d\omega)$, for all $F \in L^1(\mathscr{T}_0)$,

(Z2) $ZF \geqslant 0$, $F \geqslant 0$, $Z\mathbf{1} = \mathbf{1}$,

(Z3) $Z(FG) = ZFZG$, for all $F, G \in L^\infty(\mathscr{T}_0)$,

(Z4) $ZF(\omega, \omega') = F(\omega)$ and $ZG(\omega, \omega') = G(\omega')$ for all $F \in L^1(\mathscr{T}_0)$, $G \in L^1(\mathscr{T}_3)$

(Z5) ZF is $\mathscr{G}_0 \otimes \mathscr{R}^t$-measurable if F is $\mathscr{U}^t_{-\infty}$-measurable, for any $t \geqslant 0$.

We denote

$$p^{\omega, \omega'}(s, \mathbf{x}; t, \mathbf{y}) := (Zp^{\bullet}(s, \mathbf{x}; t, \mathbf{y}))(\omega, \omega'),$$

where $p^{\omega}(s, \mathbf{x}; t, \mathbf{y})$ is the transition of probability density corresponding to the diffusion described by (1.1).

The *transport operator* $Q: L^1(\mathscr{T}_0) \to L^1(\mathscr{T}_0)$ is defined as

$$QF(\omega') = \int_{\mathbb{R}^d} \int_\Omega p^{\omega, \omega'}(-T, -\mathbf{y}; 0, \mathbf{0})\, F(\tau_{-T, -\mathbf{y}}\omega)\, \mathbb{P}(d\omega)\, d\mathbf{y}. \tag{3.1}$$

This type of operator has been considered in the context of incompressible environments in ref. 15, cf. also ref. 16 where a version of this operator for random walks in random environments appears. We recall that Q preserves densities, i.e., for any $F \geqslant 0$ with $\int F\, d\mathbb{P} = 1$ we have $QF \geqslant 0$ with $\int QF\, d\mathbb{P} = 1$. This fact follows from the application of Proposition 3.1 below for $G \equiv 1$.

The key property of the transport operator is contained in the following proposition. Among others it explains the terminology used for Q.

Proposition 3.1. Suppose that $F \in L^1(\mathscr{T}_2)$ and G is bounded and $\mathscr{U}_0^{+\infty}$-measurable. Then, for any $t \geqslant T$ and $\psi: \mathbb{R}^d \to \mathbb{R}$ bounded and measurable

$$\int \mathbf{M}[\, G(\tau_{t, \mathbf{x}(t)}(\omega))\, \psi(\mathbf{x}(t) - \mathbf{x}(T))\,]\, F(\omega)\, \mathbb{P}(d\omega)$$

$$= \int \mathbf{M}[\, G(\tau_{t-T, \mathbf{x}(t-T)}(\omega))\, \psi(\mathbf{x}(t-T))\,]\, QF(\omega)\, \mathbb{P}(d\omega).$$

Proof. Let us denote by $\mathbf{x}^{\omega}_{s,\mathbf{x}}(t;\sigma)$, $t \geqslant s$, $(\omega,\sigma) \in \Omega \times \Sigma$ a diffusion described by (1.1) starting at time s from $\mathbf{x}$ for a given ω and let $\mathbf{z}^{\omega,\omega'}_{s,\mathbf{x}}(t) := (Z\mathbf{x}^{\bullet}_{s,\mathbf{x}}(t))(\omega,\omega')$. Also, we identify $\mathbf{x}^{\omega}_{s,\mathbf{x}}(t;\sigma) = \mathbf{x}_{s,\mathbf{x}}(t;\sigma,\omega)$. We adopt the convention of omitting the subscripts when $s = 0$, $\mathbf{x} = \mathbf{0}$.

Suppose that $N \geqslant 1$ is an integer, $0 \leqslant t_1 \leqslant \cdots \leqslant t_N$, $\mathbf{x}_1,\ldots,\mathbf{x}_N \in \mathbb{R}$, $G_1,\ldots,G_N\colon \mathbb{R} \to \mathbb{R}$ are bounded and measurable. We can write then, with $\mathbf{M}$ the expectation with respect to measure W, that

$$\begin{aligned}
&\int \mathbf{M}\left[\prod_{i=1}^{N} G_i(\mathbf{u}(t+t_i,\mathbf{x}_i+\mathbf{x}^{\omega}(t);\omega))\right]\psi(\mathbf{x}(t)-\mathbf{x}(T))\,F(\omega)\,\mathbb{P}(d\omega)\\
&\quad=\iint \mathbf{M}\left[\prod_{i=1}^{N} G_i(\mathbf{u}(t+t_i,\mathbf{x}_i+\mathbf{z}^{\omega,\omega'}(t);\omega'))\right]\\
&\qquad\times\psi(\mathbf{z}^{\omega,\omega'}(t)-\mathbf{z}^{\omega,\omega'}(T))\,F(\omega)\,\mathbb{P}(d\omega)\,\mathbb{P}(d\omega')\\
&\quad=\iiiint\left[\prod_{i=1}^{N} G_i(\mathbf{u}(t+t_i,\mathbf{x}_i+\mathbf{z}^{\omega,\omega'}(T;\sigma)\right.\\
&\qquad\left.+\mathbf{x}_{T,\mathbf{0}}(t;\sigma',\tau_{0,\mathbf{z}^{\omega,\omega'}(T;\sigma)}(\omega'));\omega'))\right]\\
&\qquad\times\psi(\mathbf{x}_{T,\mathbf{0}}(t;\sigma',\tau_{0,\mathbf{z}^{\omega,\omega'}(T;\sigma)}(\omega')))\,F(\omega)\,\mathbb{P}(d\omega)\,\mathbb{P}(d\omega')\,W(d\sigma)\,W(d\sigma').
\end{aligned}\tag{3.2}$$

The last equality following from the fact that $\mathbf{u}(t,\cdot)$ are $\mathscr{R}$ measurable for $t \geqslant T$. Using stationarity of the environment we can rewrite the utmost right hand side of (3.2) as

$$\begin{aligned}
&\iiiint\left[\prod_{i=1}^{N} G_i(\mathbf{u}(t+t_i,\mathbf{x}_i+\mathbf{x}_{T,\mathbf{0}}(t;\sigma',\tau_{0,\mathbf{z}^{\omega,\omega'}(T;\sigma)}(\omega'));\tau_{0,\mathbf{z}^{\omega,\omega'}(T;\sigma)}(\omega')))\right]\\
&\qquad\times\psi(\mathbf{x}_{T,\mathbf{0}}(t;\sigma',\tau_{0,\mathbf{z}^{\omega,\omega'}(T;\sigma)}(\omega')))\,F(\omega)\,\mathbb{P}(d\omega)\,\mathbb{P}(d\omega')\,W(d\sigma)\,W(d\sigma')\\
&\quad=\iiint_{\mathbb{R}^d}\mathbf{M}\left[\prod_{i=1}^{N} G_i(\mathbf{u}(t+t_i,\mathbf{x}_i+\mathbf{x}^{\tau_{0,\mathbf{z}}(\omega')}_{T,\mathbf{0}}(t);\tau_{0,\mathbf{z}}(\omega')))\right]\\
&\qquad\times\psi(\mathbf{x}^{\tau_{0,\mathbf{z}}(\omega')}_{T,\mathbf{0}}(t;\tau_{0,\mathbf{z}}(\omega')))\,p^{\omega,\omega'}(0,\mathbf{0};T,\mathbf{z})\,F(\omega)\,\mathbb{P}(d\omega)\,\mathbb{P}(d\omega')\,d\mathbf{z}\\
&\quad=\int\mathbf{M}\left[\prod_{i=1}^{N} G_i(\mathbf{u}(t+t_i-T,\mathbf{x}_i+\mathbf{x}(t-T)))\,\psi(\mathbf{x}(t-T))\right]QF\,d\mathbb{P} \quad\blacksquare
\end{aligned}$$

3.2. The Construction of the Invariant Measure

We start with the following.

Lemma 3.2. There exists an invariant density for Q, i.e., such $H_* \geqslant 0$ that $\int_\Omega H_* \, d\mathbb{P} = 1$ and $QH_* = H_*$. Moreover, there exist a constant $\gamma \in (0, 1)$ such that for any $F \in L^1(\mathscr{T}_2)$

$$\int_\Omega \left| Q^n F(\omega) - H_*(\omega) \int_\Omega F(\omega) \, \mathbb{P}(d\omega) \right| \mathbb{P}(d\omega) \leqslant \gamma^n, \qquad \forall n \geqslant 1. \tag{3.3}$$

γ does not depend on F.

Proof. From the classical Gaussian estimates of heat kernels, see e.g., [17, Theorem 5', p. 114 and Theorem 7', p. 116], there exist positive constants c_1 and c_2 such that for any $s < t$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $\omega, \omega' \in \Omega$

$$p^{\omega, \omega'}(s, \mathbf{x}; t, \mathbf{y}) \geqslant \frac{c_1}{(t-s)^{\frac{d}{2}}} \exp\left\{ -\frac{|\mathbf{x}-\mathbf{y}|^2}{t-s} \, c_2 \right\}.$$

Therefore, for any $F \geqslant 0$

$$\begin{aligned} QF(\omega') &= \int_{\mathbb{R}^d} \int_\Omega p^{\omega, \omega'}(-T, -\mathbf{y}; 0, \mathbf{0}) \, F(\tau_{-T, -\mathbf{y}} \omega) \, \mathbb{P}(d\omega) \, d\mathbf{y} \\ &\geqslant \frac{c_1}{T^{\frac{d}{2}}} \int_{\mathbb{R}^d} \int_\Omega \exp\left\{ -\frac{c_2 \, |y|^2}{T} \right\} F(\omega) \, \mathbb{P}(d\omega) \, d\mathbf{y} \\ &=: C(T) \int_\Omega F(\omega) \, \mathbb{P}(d\omega) \end{aligned} \tag{3.4}$$

for some positive $C(T)$. Using Theorem 5.6.2 of ref. 18 we conclude the existence of an invariant density for Q. Moreover, we conclude from (3.4) that for any $F \in L^1(\mathscr{T}_2)$ such that $\int F \, d\mathbb{P} = 0$ we have

$$QF^+ \geqslant (F^+, \mathbf{1})_{L^2} \, C(T) \qquad \text{and} \qquad QF^- \geqslant (F^-, \mathbf{1})_{L^2} \, C(T).$$

Using the fact that $(F^+, \mathbf{1})_{L^2} = (F^-, \mathbf{1})_{L^2} = 1/2 \, \|F\|_{L^1}$ we conclude that,

$$\|QF\|_{L^1} \leqslant (1 - C(T)/2) \, \|F\|_{L^1}$$

and (3.3) follows. ∎

Define $\mathbb{P}_*(d\omega) := h_*(\omega)\,\mathbb{P}(d\omega)$, where

$$h_*(\omega) := \frac{1}{T}\int_{-T}^{0}\int_{\mathbb{R}^d} p^{\omega}(t, \mathbf{y}; 0, \mathbf{0})\, H_*(\tau_{t,\mathbf{y}}\omega)\, dt\, d\mathbf{y}. \tag{3.5}$$

We show that $\mathbb{P}_*$ is a regular invariant measure. Choose $0 < t_1 < t_2 < \cdots < t_n$, $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d$ and $f_1, f_2, \ldots, f_n \colon \mathbb{R}^d \to \mathbb{R}$ bounded measurable functions. Then, for any $h \geqslant 0$,

$$\int_\Omega \prod_{i=1}^{n} f_i(\mathbf{u}(t_i+h, \mathbf{x}_i + \mathbf{x}(t_i+h)))\, \mathbb{P}_*(d\omega)$$

$$= \frac{1}{T}\int_\Omega \int_0^T \prod_{i=1}^{n} f_i(\mathbf{u}(t_i+h+s, \mathbf{x}_i + \mathbf{x}(t_i+h+s)))\, H_*(\omega)\, ds\, \mathbb{P}(d\omega). \tag{3.6}$$

Thanks to (3.3) we conclude that, the right hand side of (3.6) equals

$$\lim_{N\to\infty} \frac{1}{NT}\sum_{n=0}^{N-1}\int_\Omega\int_0^T \prod_{i=1}^{n} f_i(\mathbf{u}(t_i+h+s, \mathbf{x}_i+\mathbf{x}(t_i+h+s)))\, Q^n\mathbf{1}(\omega)\, ds\, \mathbb{P}(d\omega).$$

Proposition 3.1 allows us to rewrite this expression in the form

$$\lim_{N\to\infty} \frac{1}{NT}\sum_{k=0}^{N-1}\int_0^T\int_\Omega \prod_{i=1}^{n} f_i(\mathbf{u}(t_i+h+s+kT, \mathbf{x}_i+\mathbf{x}(t_i+h+s+kT)))\, ds\, \mathbb{P}(d\omega)$$

$$= \lim_{N\to\infty} \frac{1}{NT}\int_\Omega\int_h^{NT+h} \prod_{i=1}^{n} f_i(\mathbf{u}(t_i+s, \mathbf{x}_i+\mathbf{x}(t_i+s)))\, ds\, \mathbb{P}(d\omega)$$

$$= \lim_{N\to\infty} \frac{1}{NT}\int_\Omega\int_0^{NT} \prod_{i=1}^{n} f_i(\mathbf{u}(t_i+s, \mathbf{x}_i+\mathbf{x}(t_i+s)))\, ds\, \mathbb{P}(d\omega). \tag{3.7}$$

Repeating now calculations done in (3.6) and (3.7) in the reverse order we get that the utmost right hand side of (3.7) equals

$$\int_\Omega \prod_{i=1}^{n} f_i(\mathbf{u}(t_i, \mathbf{x}_i+\mathbf{x}(t_i)))\, \mathbb{P}_*(d\omega)$$

and the stationarity follows. It is clear that $\mathbb{P}_*$ is equivalent to $\mathbb{P}$. The only point that remains yet to be proven is ergodicity of $\mathbb{P}_* \otimes W$.

3.3. The Proof of Ergodicity

Suppose that A is a Borel subset of $C([0,+\infty);\mathbb{R})$ as specified in the definition E and $B\subseteq C([0,+\infty);\mathbb{R})$ is $\mathcal{M}_t$ measurable. Here $\mathcal{M}_t$ is the canonical filtration on $C([0,+\infty);\mathbb{R})$. Then,

$$
\begin{aligned}
&\int \mathbf{M}\mathbf{1}_A(\eta_\bullet(\omega,\cdot))\,\mathbf{1}_B(\eta_\bullet(\omega,\cdot))\,H_*(\omega)\,\mathbb{P}(d\omega)\\
&\quad=\int \mathbf{M}\mathbf{1}_A(\theta_{t+nT}(\eta_\bullet(\omega,\cdot)))\,\mathbf{1}_B(\eta_\bullet(\omega,\cdot))\,H_*(\omega)\,\mathbb{P}(d\omega)\\
&\quad=\iiint \mathbf{1}_A(\theta_{nT}(\eta_\bullet(\tau_{t,\mathbf{x}^\omega(t;\sigma)}(\omega),\sigma')))\\
&\qquad\times\mathbf{1}_B(\eta_\bullet(\omega,\sigma))\,H_*(\omega)\,\mathbb{P}(d\omega)\,W(d\sigma)\,W(d\sigma'),
\end{aligned}
\tag{3.8}
$$

with the last equality following from the strong Markov property and stationarity of the environment. Using temporal stationarity of the environment we infer that the utmost right hand side of (3.8) is equal to

$$
\begin{aligned}
&\iiint \mathbf{1}_A(\theta_{nT}(\eta_\bullet(\tau_{0,\mathbf{x}^\omega_{-t,0}(0;\sigma)}(\omega),\sigma')))\\
&\qquad\times\mathbf{1}_B(\eta_\bullet(\tau_{-t,0}(\omega),\sigma))\,H_*(\tau_{-t,0}(\omega))\,\mathbb{P}(d\omega)\,W(d\sigma)\,W(d\sigma')\\
&\quad=\iint_{\mathbb{R}} \mathbf{M}\mathbf{1}_A(\theta_{nT}(\eta_\bullet(\tau_{0,\mathbf{z}}(\omega),\cdot)))\,F_0(\omega,\mathbf{z})\,p^\omega(-t,0;0,\mathbf{z})\,\mathbb{P}(d\omega)\,d\mathbf{z},
\end{aligned}
$$

with

$$
F_0(\omega,\mathbf{z}):=\mathbf{M}[\mathbf{1}_B(\eta_\bullet(\tau_{-t,\mathbf{e}}(\omega),\cdot))\,H_*(\tau_{-t,0}(\omega))\,|\,\mathbf{x}^\omega_{-t,0}(0)=\mathbf{z}]
$$

$\mathcal{U}^0_{-\infty}\otimes\mathcal{B}(\mathbb{R})$-measurable. Using stationarity of the environment in the $\mathbf{z}$ variable we obtain that the left hand side of (3.8) equals

$$
\int \mathbf{M}\mathbf{1}_A(\theta_{nT}(\eta_\bullet(\omega,\cdot)))\,F(\omega)\,\mathbb{P}(d\omega),
$$

with

$$
F(\omega):=\int_{\mathbb{R}} F_0(\tau_{0,-\mathbf{z}}(\omega),\mathbf{z})\,p^{\tau_{0,-\mathbf{z}}(\omega)}(-t,0;0,\mathbf{z})\,d\mathbf{z}
$$

$\mathscr{U}^{0}_{-\infty}$-measurable. Applying Proposition 3.1 we conclude that the utmost right hand side of (3.8) equals

$$\int \mathbf{M}\mathbf{1}_A(\eta_\bullet(\omega,\cdot))\, Q^n F(\omega)\, \mathbb{P}(d\omega). \tag{3.9}$$

Letting $n \uparrow +\infty$ we obtain, thanks to Lemma 3.2,

$$\int \mathbf{M}\mathbf{1}_A(\eta_\bullet(\omega,\cdot))\, \mathbf{1}_B(\eta_\bullet(\omega,\cdot))\, H_*(\omega)\, \mathbb{P}(d\omega)$$

$$= \int \mathbf{M}\mathbf{1}_A(\eta_\bullet(\omega,\cdot))\, H_*(\omega)\, \mathbb{P}(d\omega) \int \mathbf{M}\mathbf{1}_B(\eta_\bullet(\omega,\cdot))\, H_*(\omega)\, \mathbb{P}(d\omega)$$

for any $B \in \mathscr{M}_t$, $t \geqslant 0$. Hence

$$\int \mathbf{M}\mathbf{1}_A(\eta_\bullet(\omega,\cdot))\, H_*(\omega)\, \mathbb{P}(d\omega) = \left[\int \mathbf{M}\mathbf{1}_A(\eta_\bullet(\omega,\cdot))\, H_*(\omega)\, \mathbb{P}(d\omega) \right]^2,$$

which, thanks to strict positivity of H_*, proves that $\mathbb{P}_* \otimes W[\eta_\bullet \in A] = 0$ or 1. ▮

4. THE PROOF OF THEOREM 2.3

Let

$$d_{i,j}(t) := \frac{1}{t} \int_0^t \int_0^s \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))]\, ds\, ds_1,$$

$$a_{i,j}(t) := \frac{1}{t} \int_0^t \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, w_j(s)]\, ds,$$

where $\tilde{\mathbf{u}} = (\tilde{u}_1,..., \tilde{u}_d) := \mathbf{u} - \mathbf{v}_*$, with $\mathbf{v}_* = (v_{1,*},..., v_{d,*})$ the Stokes' drift defined in Corollary 2.2. Using Itô formula we obtain

$$\mathbb{E}\mathbf{M}\left[\frac{(x_i(t) - v_{i,*}t)(x_j(t) - v_{j,*}t)}{t} \right]$$

$$= d_{i,j}(t) + d_{j,i}(t) + \sqrt{2\kappa}\, (a_{i,j}(t) + a_{j,i}(t)) + \kappa\delta_{i,j}.$$

The first part of the theorem is therefore a conclusion from the following lemma.

Lemma 4.1. The following limits exist

$$a_{i,j} := \lim_{t\uparrow+\infty} a_{i,j}(t), \tag{4.1}$$

$$d_{i,j} := \lim_{t\uparrow+\infty} d_{i,j}(t), \qquad i,j=1,\ldots,d. \tag{4.2}$$

Proof of (4.1). We prove first that $\lim_{n\uparrow+\infty} a_{i,j}(nT)$ exists for each $i,j=1,\ldots,d$. Note that

$$\begin{aligned} a_{i,j}(nT) &= \frac{1}{nT}\int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(s)]\, ds \\ &+ \frac{1}{nT}\int_T^{nT} \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, [w_j(s)-w_j(T)]]\, ds \\ &+ \frac{1}{nT}\int_T^{nT} \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(T)]\, ds. \end{aligned} \tag{4.3}$$

The middle term on the right hand side of (4.3) equals, thanks to Proposition 3.1 and spatial homogeneity of the field,

$$\frac{1}{nT}\int_0^{(n-1)T} \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(s)\, Q^k \mathbf{1}]\, ds.$$

Repeating this procedure n times we arrive at

$$\begin{aligned} a_{i,j}(nT) &= \frac{1}{nT}\sum_{k=0}^{n}\int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(s)\, Q^k\mathbf{1}]\, ds \\ &+ \frac{1}{nT}\sum_{k=0}^{n-1}\int_T^{(n-k)T} \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(T)\, Q^k\mathbf{1}]\, ds. \end{aligned}$$

Using Lemma 3.2 we obtain that $a_{i,j}(nT)$ is, up to a term of order $O(n^{-1/2})$, equal to

$$\begin{aligned} &\int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(s)\, H_*]\, ds \\ &+ \frac{1}{nT}\sum_{k=0}^{n-1}\int_T^{(n-k)T} \mathbb{E}\mathbf{M}[\tilde{u}_i(s,\mathbf{x}(s))\, w_j(T)\, H_*]\, ds. \end{aligned} \tag{4.4}$$

However, for any integer $r \geqslant 1$ and $s \in [rT, (r+1)\,T]$

$$\mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, w_j(T)\, H_*] = \mathbb{E}\mathbf{M}[\tilde{u}_i(s-rT, \mathbf{x}(s-rT))\, Q^{r-1} Y_j]$$

for

$$Y_j(\omega) := \int_{\mathbb{R}^d} Y_j^{(0)}(\mathbf{y}, \tau_{-T, -\mathbf{y}}(\omega))\, p^{\omega}(-T, -\mathbf{y}; 0, \mathbf{0})\, d\mathbf{y},$$

with $Y_j^{(0)}(\mathbf{y}, \omega) := \mathbf{M}[w_j(T)\, H_*(\omega) \mid \mathbf{x}(T) = \mathbf{y}]$. With

$$Z_j := \sum_{r=0}^{+\infty} Q^r \left(Y_j - H_* \int Y_j\, d\mathbb{P} \right). \tag{4.5}$$

We can write that in the limit as $n \uparrow +\infty$ the expression in (4.4) equals

$$\int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, w_j(s)\, H_*]\, ds + \int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, Z_j]\, ds$$

and (4.1) follows. We have used here the fact that $\int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, H_*]\, ds = 0$.

Proof of (4.2). As in the previous part of the proof it suffices to show that the limit $\lim_{n \uparrow +\infty} d_{i,j}(nT)$ exists for each $i, j = 1,..., d$. Note that $d_{i,j}(nT) = I_n + II_n$, where

$$I_n := \frac{1}{nT} \sum_{k=1}^{n} \sum_{m=1}^{k-1} \int_{(k-1)\,T}^{kT} \int_{(m-1)\,T}^{mT} \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))]\, ds\, ds_1$$

and

$$II_n := \frac{1}{nT} \sum_{k=1}^{n} \int_{(k-1)\,T}^{kT} \int_{(k-1)\,T}^{s} \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))]\, ds\, ds_1.$$

Thanks to Proposition 3.1 we can rewrite I_n as being equal to

$$\frac{1}{nT} \sum_{m=1}^{n-1} \sum_{k=m+1}^{n} \int_{(k-m-1)\,T}^{(k-m)\,T} \int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))\, Q^{m-1}\mathbf{1}]\, ds\, ds_1. \tag{4.6}$$

Using Lemma 3.2 we conclude that the expression in (4.6) equals, up to a term of magnitude $o(1)$, when $n \uparrow +\infty$,

$$\frac{1}{nT} \sum_{m=1}^{n-1} \sum_{k=m+1}^{n-1} \int_{(k-m)\,T}^{(k-m+1)\,T} \int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))\, H_*]\, ds\, ds_1. \tag{4.7}$$

Let $Y_j := \int_0^T \tilde{u}_j(s_1, \mathbf{x}(s_1))\, H_*\, ds_1$ and $Z_j(\omega, \mathbf{y}) := \mathbf{M}[Y_j \,|\, \mathbf{x}(T) = \mathbf{y}]$. Note that $Z_j\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ is $\mathscr{U}_0^T \otimes \mathscr{B}(\mathbb{R}^d)$-measurable.

We can rewrite (4.7) as being equal to

$$\frac{1}{nT} \sum_{m=1}^{n-1} \sum_{k=0}^{n-m-1} \int_{kT}^{(k+1)T} \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{Y}_j]\, ds, \tag{4.8}$$

with

$$\tilde{Y}_j(\omega) := \int_{\mathbb{R}^d} p^{\omega}(-T, \mathbf{y}; 0, \mathbf{0})\, Z_j(\tau_{-T, \mathbf{y}}(\omega), \mathbf{y})\, d\mathbf{y}, \tag{4.9}$$

a $\mathscr{U}^0_{-\infty}$-measurable random variable. Thanks to Lemma 3.2 we conclude that

$$\begin{aligned} \lim_{n \uparrow +\infty} I_n &= \lim_{n \uparrow +\infty} \frac{1}{nT} \sum_{m=1}^{n-1} \sum_{k=0}^{n-m-1} \int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, Q^k \tilde{Y}_j]\, ds \\ &= \int_0^T \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, Z_j]\, ds, \end{aligned} \tag{4.10}$$

with $Z_j := \sum_{k=0}^{+\infty} Q^k \tilde{Y}_j$.

On the other hand, II_n equals to

$$\frac{1}{nT} \sum_{k=1}^{n} \int_0^T \int_0^s \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))\, Q^{k-1}\mathbf{1}]\, ds\, ds_1.$$

which, as $n \uparrow +\infty$, tends to

$$\int_0^T \int_0^s \mathbb{E}\mathbf{M}[\tilde{u}_i(s, \mathbf{x}(s))\, \tilde{u}_j(s_1, \mathbf{x}(s_1))\, H_*]\, ds\, ds_1. \quad \blacksquare$$

APPENDIX A. THE FACTORIZATION OF THE VELOCITY FIELD

Suppose that $\mathbf{u}(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is a time space stationary random field over $\mathscr{T}_0$ and let $\mathscr{U}_t$ be the natural filtration of σ-algebras corresponding to $\mathbf{u}$, i.e., the σ-algebra $\mathscr{U}_t$ is generated by $\{\mathbf{u}(s, \mathbf{x}), \mathbf{x} \in \mathbb{R}^d, s \leqslant t\}$. Let $\mathscr{U}^t$ be the σ-algebra generated by $\{\mathbf{u}(t, \mathbf{x}), \mathbf{x} \in \mathbb{R}^d, t \leqslant s\}$. Assume also that the assumption FDT holds.

Since $\mathscr{U}_t$ are countably generated σ-algebras for all $t \in \mathbb{R}$ (in the sense of definition given in [19, p. 63], there exist random variables ξ_t, such that

$\underline{\mathscr{U}_t = \sigma(\xi_t)} \vee \mathscr{N}$ for all $t \in \mathbb{R}$. Recall here that $\mathscr{N}$ is the σ-ring of $\mathbb{P}$-null sets in $\mathscr{B}(\Omega)$. Set

$$\Phi_{\xi_t}[A \mid \xi_0(\omega)] := \mathbb{P}[[\xi_t \in A] \cup N \mid \mathscr{U}_0](\omega), \qquad \text{for any} \quad A \in \mathscr{B}(\mathbb{R}), \quad N \in \mathscr{N}.$$

Define the conditional probability measures $\mathbb{P}[\,\cdot \mid \mathscr{U}_0] := \Phi_{\xi_t}[\,\cdot \mid \xi_0]$

We assume that

(AC) for any finite set $I \subset \mathbb{Q} \times \mathbb{Q}^d$ the random variables $\mathbf{u}(t, \mathbf{x}; \cdot)$, $(t, \mathbf{x}) \in I$ have absolutely continuous distributions with respect to the product of $\#I$ copies of the Lebesgue measures. Here $\mathbb{Q}$ denotes the set of rational numbers.

Suppose that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are sub σ-algebras of $\overline{\mathscr{B}(\Omega)}$. We say that $\mathscr{B}$ and $\mathscr{C}$ factor $\mathscr{A}$, which is denoted by $\mathscr{A} = \mathscr{B} \otimes \mathscr{C}$, if $\mathscr{A}$ is generated by $\mathscr{B}$ and $\mathscr{C}$ and they are independent.

Lemma A.1. Under the assumption AC $(\mathscr{U}_{-\infty}^t)$ admits a factorization, i.e., there exists a filtration of σ-algebras $\mathscr{R}^t$, $t \geqslant 0$ such that for any $t \geqslant 0$ we have $\mathscr{U}_{-\infty}^t = \mathscr{U}_{-\infty}^0 \otimes \mathscr{R}^t$. In addition, if FDT holds then there exists $T > 0$, for which $\mathscr{U}_T^{+\infty} \subset \mathscr{R}$, where $\mathscr{R} := \bigvee_{t=0}^{\infty} \mathscr{R}^t$.

Proof. By virtue of Theorem 5 of ref. 19 we know that $\mathscr{V}_0$ factors $\mathscr{V}_t$, $(t > 0)$ provided that the conditional probabilities $\mathbb{P}[\,\cdot \mid \mathscr{V}_0](\omega)$ are atomless on $\mathscr{V}_t$, $\mathbb{P}$-a.s. in ω. Recall here that a set $A \in \mathscr{S}$, where $(\Xi, \mathscr{S}, \mu)$ is a certain measure space, is called an *atom*, if for any $B \in \mathscr{S}$, $B \subset A$, such that $\mu[B] < \mu[A]$, $\mu[B] = 0$. Assume that $\mathbb{P}[\,\cdot \mid \mathscr{V}_0](\omega)$ possesses atoms in $\mathscr{V}_t$ on a set of ω-s having a positive measure $\mathbb{P}$. For some $N \geqslant 1$ there exists a set $L \in \mathscr{V}_0$ such that the conditional distribution function $F_{\xi_t}(x \mid \xi_0(\omega)) := \Phi_{\xi_t}[\xi_t < x \mid \xi_0(\omega)]$, $x \in \mathbb{R}$ has a jump of size not less than $\frac{1}{N}$ in the interval $[-N, N]$ for $\mathbb{P}$-a.s. $\omega \in L$. Let $f(\omega)$ be the minimum of such jump sites in $[-N, N]$ for a fixed ω. Then $A := [\omega \in L{:}\ \xi_t(\omega) = f(\omega)]$ is an atom with $\mathbb{P}[A \mid \mathscr{V}_0](\omega) \geqslant \frac{1}{N}$ for $\mathbb{P}$-a.s. ω. Let $\mathscr{V}_t^{(k)}$ denote the sub-σ-algebra of $\mathscr{V}_t$ generated by the sites $|\mathbf{x}| \leqslant k$, $\mathbf{x} \in (\frac{1}{2^k}\mathbb{Z})^d$. Thanks to AC, $\mathbb{P}[\,\cdot \mid \mathscr{V}_0^{(k)}]$ defined on $\mathscr{V}_t^{(m)}$ are atomless for any $k, m \geqslant 1$, $\mathbb{P}$-a.s. Let $\mathscr{A}_k$ be a countable algebra of sets such that $\mathscr{V}_t^{(k)} = \mathscr{A}_k \vee \mathscr{N}$ and $\mathscr{A} := \bigvee_{k=1}^{\infty} \mathscr{A}_k \vee \{A, A^c\}$, with $A^c := \Omega \setminus A$.

The martingale a.s. convergence theorem yields that

$$\lim_{k \to \infty} \mathbb{P}[C \mid \mathscr{V}_0^{(k)}] = \mathbb{P}[C \mid \mathscr{V}_0] \qquad \text{for all} \quad C \in \mathscr{A}, \quad \mathbb{P}\text{-a.s.}$$

For an arbitrary $\varepsilon > 0$ there exists $\tilde{A} \in \mathscr{V}_t^{(m)}$ such that $\mathbb{P}[A \triangle \tilde{A}] \leqslant \varepsilon$. Then

$$\mathbb{P}[A \triangle \tilde{A}] = \int \mathbb{P}[A \triangle \tilde{A} \mid \mathscr{V}_0] \, d\mathbb{P} \geqslant \int \liminf_{k \to \infty} \mathbb{P}[A \triangle \tilde{A} \mid \mathscr{V}_0^{(k)}] \, d\mathbb{P}.$$

Choosing a suitable positive ε (for example $\varepsilon = \frac{1}{1000N}$), for an $\omega \in L$ we can find a sequence $(k_l) \subset (k)$ such that

$$\mathbb{P}[A \triangle \tilde{A} \mid \mathscr{V}_0^{(k_l)}](\omega) \leqslant \frac{1}{128N}.$$

Measures $\mathbb{P}[\,\cdot \mid \mathscr{V}_0^{(k_l)}](\omega)$ are atomless on $\mathscr{V}_t^m$ $(m \geqslant 1)$. Therefore one can find $B_1^{(1)}, B_1^{(2)} \in \mathscr{V}_t^{(m)}$ such that $B_1^{(1)} \cap B_1^{(2)} = \varnothing$, $B_1^{(1)}, B_1^{(2)} \subset \tilde{A}$ and

$$\frac{1}{16N} < \mathbb{P}[B_1^{(i)} \mid \mathscr{V}_0^{(k_1)}](\omega) < \frac{1}{8N}, \qquad i = 1, 2.$$

We can construct also descending families of sets $(B_p^{(i)})$, $p \geqslant 1$, $i = 1, 2$ such that

$$\frac{1}{16N} < \mathbb{P}[B_p^{(i)} \mid \mathscr{V}_0^{(k_l)}](\omega) < \frac{1}{8N}, \qquad i = 1, 2, \quad l = 1,..., p.$$

Set $B^{(i)} := \bigcap^{p \geqslant 1} B_p^{(i)}$, $i = 1, 2$. Then

$$\frac{1}{16N} < \lim_{p \to \infty} \mathbb{P}[B_p^{(i)} \mid \mathscr{V}_0^{(k_l)}](\omega) = \mathbb{P}[B^{(i)} \mid \mathscr{V}_0^{(k_l)}](\omega), \qquad i = 1, 2.$$

This implies that, in fact,

$$\frac{1}{16N} < \mathbb{P}[B_p^{(i)} \mid \mathscr{V}_0^{(k_l)}](\omega) < \frac{1}{8N}, \qquad \forall l, p \in \mathbb{N}, \quad i = 1, 2.$$

Thus, for all $p \in \mathbb{N}$

$$\frac{1}{16N} \leqslant \mathbb{P}[B_p^{(i)} \mid \mathscr{V}_0](\omega), \qquad i = 1, 2,$$

and, in consequence,

$$\frac{1}{16N} \leqslant \mathbb{P}[B^{(i)} \mid \mathscr{V}_0](\omega), \qquad i = 1, 2.$$

For $i = 1, 2$ put $A_i := B^{(i)} \cap A$. We have $\mathbb{P}[A_i \mid \mathscr{V}_0](\omega) \geqslant \frac{1}{32N}$ and $A_1 \cap A_2 = \varnothing$. This contradicts the assumption that A is an atom. ∎

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